Reflections on symmetry and proof

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This article uses notions of symmetry to approach the solutions to a broad range of mathematical problems. It responds to Krutetskii's criteria for mathematical ability as well as the outcomes which guide the Extension 1 & 2 Mathematics courses of the Board of Studies NSW.

The concept of symmetry is fundamental, indeed foundational, to mathematics. Arguments and proofs based on symmetry are often aesthetically pleasing because they are subtle and succinct and non-standard. By symmetry, the person in the street usually means the exact correspondence in size and position of opposite parts, seen as an equal distribution of parts across a dividing line or about a centre. It is considered to be an attribute either of the whole or of the parts composing it. A more subtle appreciation of the term symmetry takes into account a harmony of parts with each other and with the whole, seen as a mutual relation of parts, as a fitting, regular or balanced arrangement and relation of parts or elements. (Onions, 1978).

Non-standard approaches to mathematical reasoning may well demand a wide range of cognitive abilities, and take us out of the realms of "working mathematically" that we are used to in our classrooms. Indeed, the psychology of mathematical ability has been explored by Krutetskii. His exciting research, first published in Russian in 1963, seems to be little known. For Krutetskii (1976, pp. 84–88), mathematical ability is seen in terms of a student's ability

- to formalise;
- to symbolise;
- to generalise;
- to carry out sequential deductive logic;
- to syncopate or to curtail logic or argument;
- to reverse logical thinking or find the converse;
- to be flexible in mathematical methods used;
- to conceptualise spatially; and
- to develop before puberty a "mathematical mind."

Students with high ability or high potential in mathematics enjoy and express these abilities in a way which is markedly and qualitatively differenti-

ated from the ability of typical age peers, and which is measurable in their ability to solve problems. For Krutetskii, the ramifications of this are two-fold. On the one hand, we are able to assess mathematical aptitude through problem solving activities that respond to these abilities, or at least the first eight of them. On the other hand, mathematical ability can be fostered, nay, developed, by scaffolding a student through a series of problems that address or invite the use of these abilities when approaching their solution (Krutetskii, after all, was a good student of Vygotsky). The problems that Krutetskii uses for both of these processes are very interesting because of their nature, scope and level of difficulty, and are worth seeking out.

The following nine problems, or rather their solutions using various notions of symmetry, are intended to illustrate "Krutetskii's abilities." Accordingly, they could have two uses. First, they reflect the need for both standard and non-standard approaches to the teaching of mathematical concepts in the classroom: the mathematical abilities need to be modelled and scaffolded, and alternative approaches to working mathematically need to be recognised and encouraged and honoured. Second, they suggest that to more adequately assess high potential in mathematics, methods more dynamic than traditional classroom approaches to measuring mathematical ability need to be employed.

A warm-up exercise

Imagine a long rectangular strip of paper. Visualise tying an overhand knot in this strip of paper. The knot will fit snugly together by gently jiggling the paper into a firm, flat knot and neatly creasing the paper where necessary. What shape is formed by the knot?

Pons asinorum

One of the first formal results met by mathematics students is the proof of the theorem that the "base" angles of an isosceles triangle are equal. The obvious method is to add a construction line and to prove that the two "halves" are congruent. A more subtle approach is to follow the hint given by Pappus of Alexandria (circa AD 340), who used the intrinsic symmetry of the isosceles triangle in an instructive way. His argument may be conceptually easier to follow if we use the symmetry of the isosceles triangle along with the symmetry of a reflection. Consider ΔABC reflected in the line l to produce $\Delta A'C'B'$ (see Figure 1). Now, since $AB = A'C'$ and $AC = A'B'$, and since $\angle BAC = \angle C'A'B'$, not only are the two triangles congruent, which they are thanks to the reflection, but also, and more importantly, $\Delta ABC \equiv \Delta A'C'B'$. Hence $\angle B = \angle C' = \angle C$. (Dodgson (Lewis Carroll), 1879, p. 48.)

The proof of Pappus (Coxeter, 1969, p. 6) simply considers ΔABC with $AB = AC$, notes that $\Delta ABC \equiv \Delta ACB$, and concludes that $\angle B = \angle C$, οπερ εδει δαιξει.

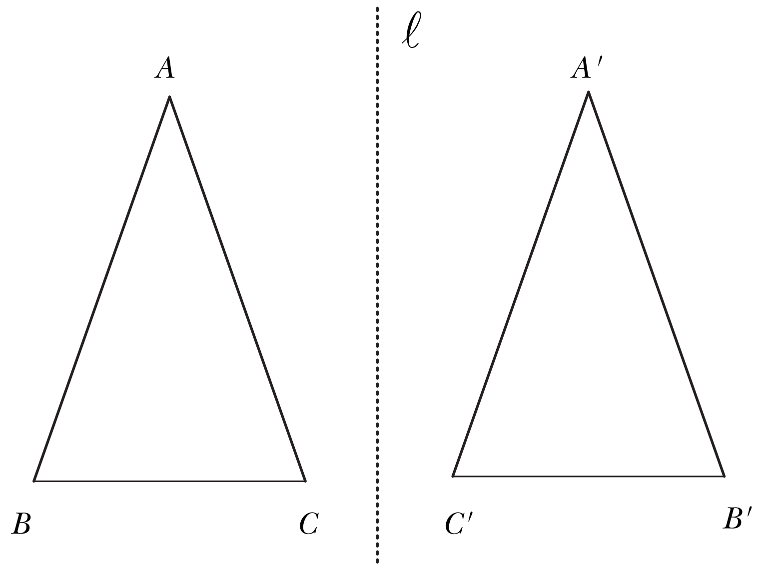

Figure 1

A chessboard problem

Lack of symmetry can also serve as a useful tool for solving problems. Consider a normal 8 × 8 chessboard and consider rectangular dominoes that are the same size as two of the squares of the chessboard joined together along one edge. Now remove two squares from diagonally opposite corners of the chessboard. Is it possible to exactly cover the remaining shape of 62 squares with 31 dominoes (see Figure 2)?

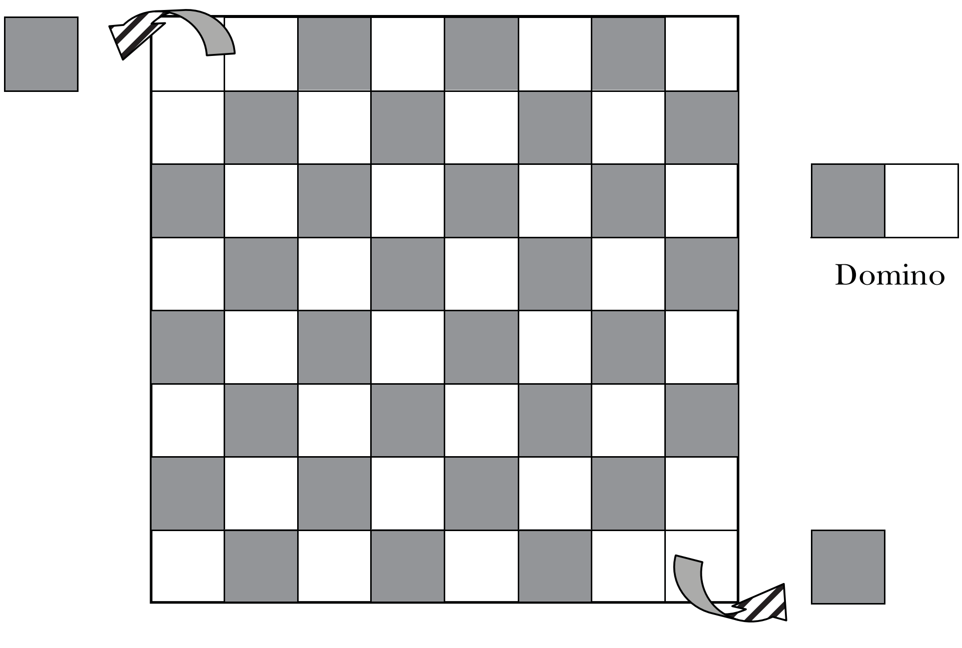

Figure 2

We may use the symmetry of the domino and the asymmetry of the mutilated chessboard to find a simple solution to this problem. By this, I mean with respect to the squares on the chessboard. Make sure, first, that the squares of the chessboard are, as usual, alternately coloured black and white. Now note that the two squares removed are the same colour, so that the number of black squares remaining does not equal the number of white squares. Also note that each domino will necessarily cover one black square and one white square. Therefore the desired covering is impossible.

A calculus problem?

The very mention of symmetry gives away a neat approach to this apparent calculus problem. Imagine the point P which lies 7 units due west of a straight fence f which runs due north–south, and imagine the point Q which lies 5 units due west of the fence and which is 5 units south (and 2 units east) of P. We have to travel from P to Q, but along the way we have to visit the fence (see Figure 3). The problem is to find the shortest distance needed to do this.

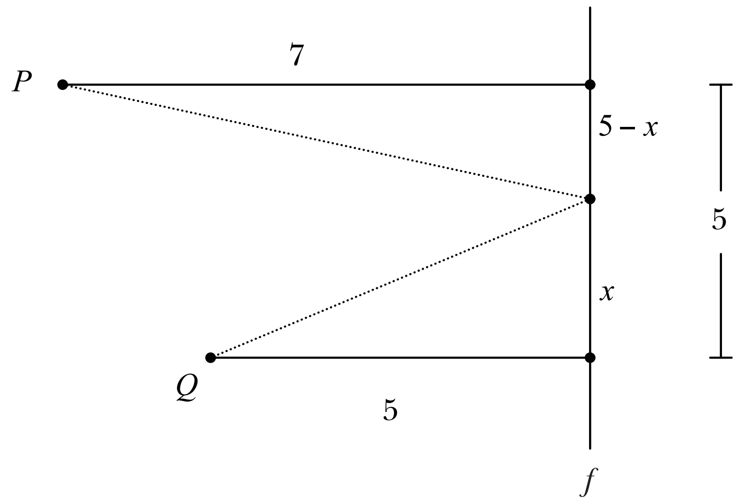

Figure 3

One approach is to make careful use of differential calculus to minimise

$$y = \sqrt{25 + x^2} + \sqrt{74 - 10x + x^2}$$

A neater approach is to use the symmetry of reflection. Since we have to visit the fence, the situation is the same if Q is on the other side of the fence, so consider the point Q' which is the reflection of Q in the line f. The shortest distance between P and Q, visiting the fence, is the same as the shortest distance between P and Q'; this, of course, is a straight line. The rest follows from a simple application of Pythagoras' Theorem, and, to be sure, the numbers have been judiciously chosen to give a Pythagorean triad.

Symmetry and conics

The "shortest path property of reflection" also forms the basis of a beautiful proof, discovered by Heron *circa* AD 100 (Stillwell, 2002, pp. 30–31), of the reflection or "whispering" property of an ellipse. This property states that a tangent t to an ellipse at a point P makes equal angles with the lines joining P to the foci E and F (see Figure 4).

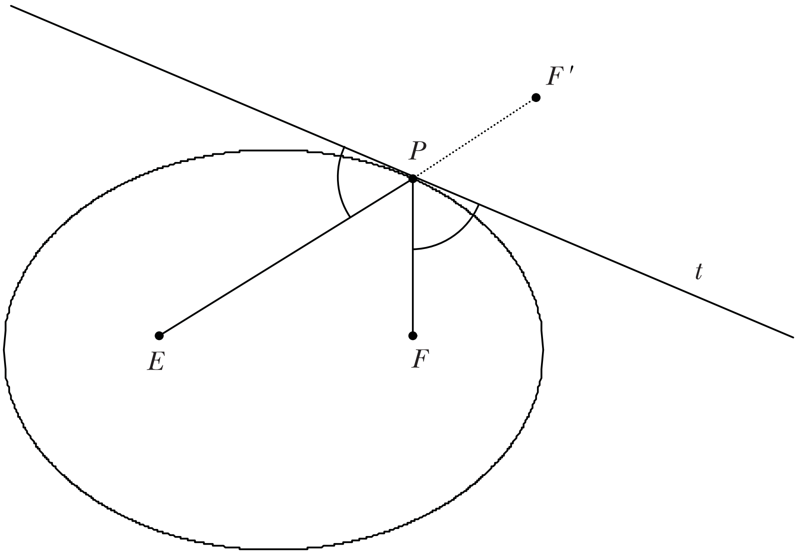

Figure 4

The standard approach is to use coordinate geometry and

$$\tan(A-B)=\frac{m_1-m_2}{1+m_1.m_2}$$

but the equations do turn out to be a tad messy. The best, I think, of these methods considers the unit vectors $\mathbf{v}_E$ and $\mathbf{v}_F$ on the focal radii towards P, and a normal vector $\mathbf{n}$ to the ellipse at P. Here, fortunately, it is not too difficult to show that

$$\mathbf{n}\cdot\mathbf{v}_E=\mathbf{n}\cdot\mathbf{v}_F,$$

and hence the cosine of the angles are equal, from which the result follows (Hansen, 1998, pp. 15–17).

Heron's idea, however, is to consider the point F' which is the reflection of F in the tangent t. First, the shortest distance from E to t to F is via the point P. This is easy to see, because we know that distance EPF is constant for all points P on the ellipse, and all other points on t lie outside the ellipse. Second, the shortest distance from E to F' is a straight line, and, since $FP = F'P$, it follows that P must lie on the straight line EF'. Third, matching up equal angles completes the proof of the property.

A wondrous fact about conic sections is that most properties have taxonomic significance; that is, there is an intrinsic symmetry between the ellipse, the parabola and the hyperbola. So it is natural to ask whether the whispering property for the ellipse also holds for the parabola. Indeed, a parabola is really an ellipse with one of the foci at infinity, and the reflection property of a parabola is well known. Following the wording above, this property may be

restated: a tangent t to a parabola at a point P makes equal angles with the line joining P to the focus S and the line through P parallel to the axis of the parabola (see Figure 5).

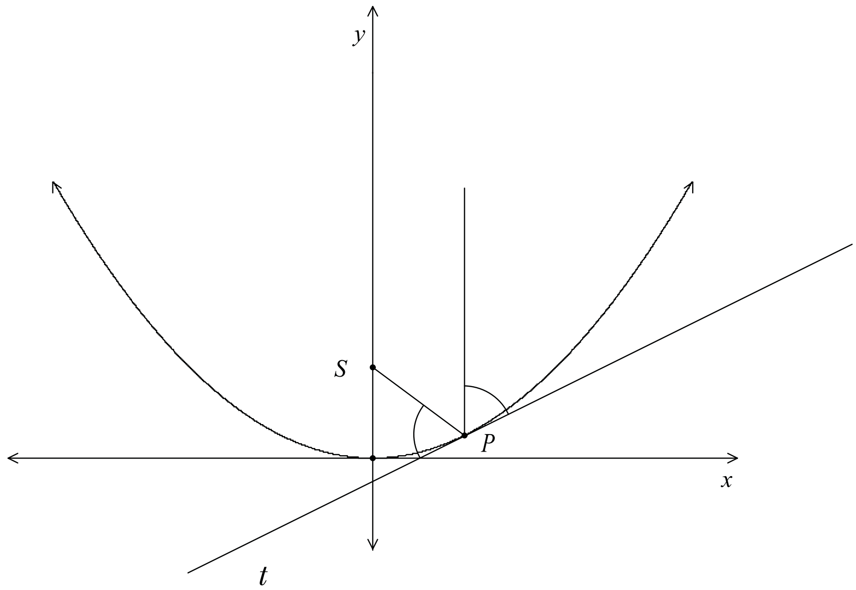

Figure 5

At first glance, a proof of this result using Heron's idea appears to be intractable. The reflection $S'(x', y')$ of the point $S(x, y)$ in the line t is given by:

$$x' = x \cos 2\theta + y \sin 2\theta + 2l \cos \varphi$$
$$y' = x \sin 2\theta - y \cos 2\theta + 2l \sin \varphi$$

where θ is the angle of inclination of the line, l is the length of the perpendicular from the origin O to the line t, and, if $l \neq 0$, φ is the angle of the vector represented by this directed perpendicular (Gans, 1969, pp. 56–58).

Filling in some details shows a focus $S(0, a)$, a point $P(2ap, ap^2)$ on the parabola, and, at P, a tangent t whose equation is given by $y = px - ap^2$. If everything is all for the best in the best of all possible worlds, the reflection of S in t will be $S'(2ap, -a)$. A quick check confirms that the midpoint $M\ (ap, 0)$ of S and S' lies on t; that is, not only is $\Delta SPS'$ isosceles but t is also its axis of symmetry. Matching up equal angles completes the proof of the property.

If you think that the Cartesian geometry is ungainly, it may be more aesthetically pleasing to use symmetry in a different sense and to remember that one definition of a parabola is the locus of points equidistant from a point (the focus) and a line (the directrix). This may be demonstrated by taking a blank piece of paper, placing an ink dot S a few centimetres from an edge, and folding this edge over so that it touches S at the point S'. The fold is part of the envelope of the parabola and is a tangent t to the parabola at some point P. It is obvious that $\Delta SPS'$ is isosceles, and also that the midpoint M of S and S' lies on t. The result immediately follows.

A possible extension exercise is to explore the equivalent property for a hyperbola (with the caveat that the world may have been turned inside out).

A projectile problem

This next problem comes from the end of a three unit (now Extension 1) HSC mathematics examination (Board of Studies NSW, 2000, Question 7(b)iv). Figure 6 shows an inclined plane, which makes an angle of α radians with the horizontal. A projectile is fired from O, at the bottom of the incline, with a speed of $V\,\mathrm{ms}^{-1}$ at an angle of elevation θ to the horizontal, as shown.

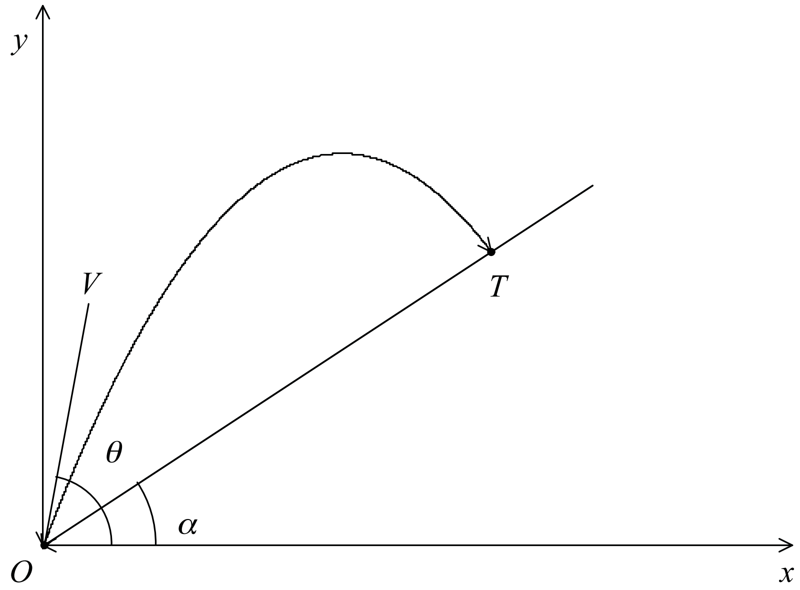

Figure 6

Along the way, we have assumed that $2V^2/g = 1$, where g is the acceleration due to gravity, to produce a simplified equation of the trajectory

$$y = x \tan\theta - x^2 \sec^2\theta$$

Either solving simultaneous equations or using polar coordinates then allows us to show that the range r of the projectile up the inclined plane is

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2\alpha}$$

The maximum range R is easy to find by remembering that

$$\sin A.\cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$

so that

$$r = \frac{\sin(2\theta - \alpha) - \sin\alpha}{2\cos^2\alpha}$$

This is clearly maximum when $\sin(2\theta - \alpha) = 1$, which means that

$$R = \frac{1}{2(1 + \sin\alpha)}$$

For future reference, note that this occurs when

$$2\theta - \alpha = \frac{\pi}{2}$$

Now we are asked to consider the trajectory of the projectile for which this maximum range R is achieved. The problem is to show that, for this trajectory, the initial direction is perpendicular to the direction at which the projectile hits the inclined plane.

The standard approach is to find $\dfrac{dy}{dx}$ at O and T. If you prefer to use $\dfrac{\dot{y}}{\dot{x}}$, it may be handy to use the t-results to simplify the expressions obtained. In either case, it takes about a page of solid working out to get the desired result, and the gentle reader is invited to confirm this.

However, we may use the intrinsic symmetry of the trajectory to find an immediate solution to the problem. Fire the projectile backwards from T, at some speed V' ms^{-1} and at an angle of φ to the horizontal, so that it lands at its maximum range at O. Use this symmetry to redraw the diagram, reflecting in the y-axis, shifting the axis so that the origin O' is at T, and relabelling O as T' (see Figure 7).

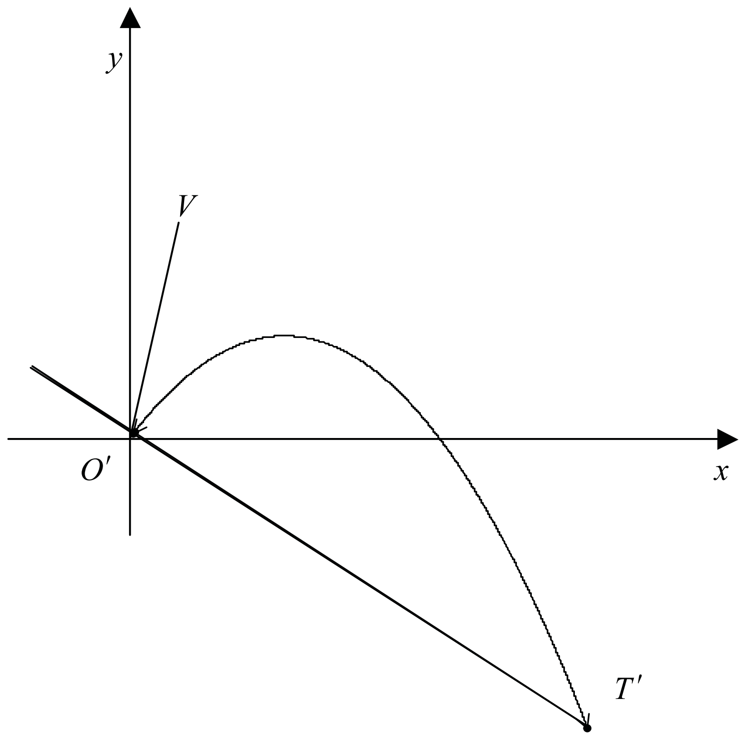

Figure 7

Recall that R occurs when $$2\theta-\alpha=\frac{\pi}{2}$$

that is, when $$\theta=\frac{\pi}{4}+\frac{\alpha}{2}$$

and hence, *mutatis mutandis,* also when
$$\varphi=\frac{\pi}{4}-\frac{\alpha}{2}$$

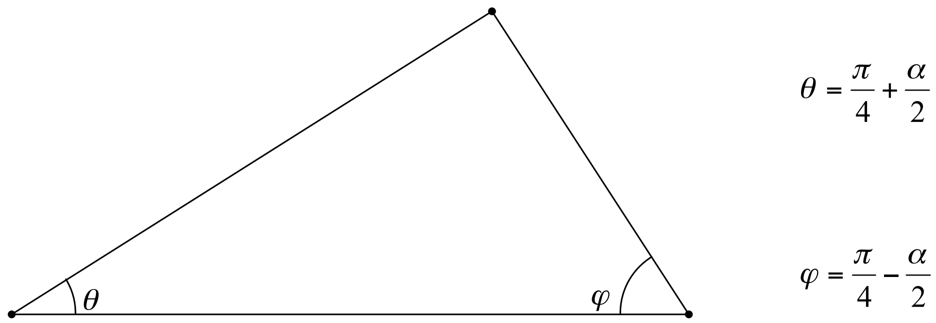

Figure 8

Now note with glee that

$$\theta + \varphi = \frac{\pi}{2}$$

and a final glimpse of Figure 8 will reveal that we have found the desired result.

A number theory problem

Let us now consider

$$\left(1+\sqrt{2}\right)^n = p_n + q_n\sqrt{2} \qquad (*)$$

where p_n and q_n are integers.

The problem is to show that p_n is the integer nearest to $q_n\sqrt{2}$.

At first sight, this problem appears to be beyond the realms of a secondary mathematics course, but here we may capitalise on the symmetry evidently intrinsic to $(1 + \sqrt{2})^2$ and $(1 - \sqrt{2})^2$. Note that we may indeed write

$$(1 - \sqrt{2})^n = p_n - q_n\sqrt{2}$$

where p_n and q_n have exactly the same values as in (*) above.

Finally, we need simply remark that

$$\left|1-\sqrt{2}\right| < \frac{1}{2}$$

and we are done.

The theorem of Pythagoras

Loomis (1968) draws attention to 370 different proofs of Pythagoras' Theorem, and doubtless there have been many more since the book was first published in 1940. Notions of symmetry, in several senses of the word, offer a new proof of the theorem. If I may be so bold, it could be suggested that this proof is even shorter than Legendre's solution (Loomis, 1968, pp. 23–24).

Taking an idea from Eudoxus on proportion (Eves, 1980, pp. 53–61), the theorem of Pythagoras becomes a corollary of the more general theorem: "the area of the shape on the hypotenuse is equal to the sum of the areas of the similar shapes on the other two sides." Using the same idea, the general

theorem will follow if it can be shown to hold for a specific case. In particular, choose for the shapes triangles that are similar to the original right-angled triangle. Rather than drawing the shapes on the sides of the right-angled triangle and facing outwards, as is usually done with the squares, the similar triangles may be drawn facing inwards as in Figure 9. The three results immediately follow.

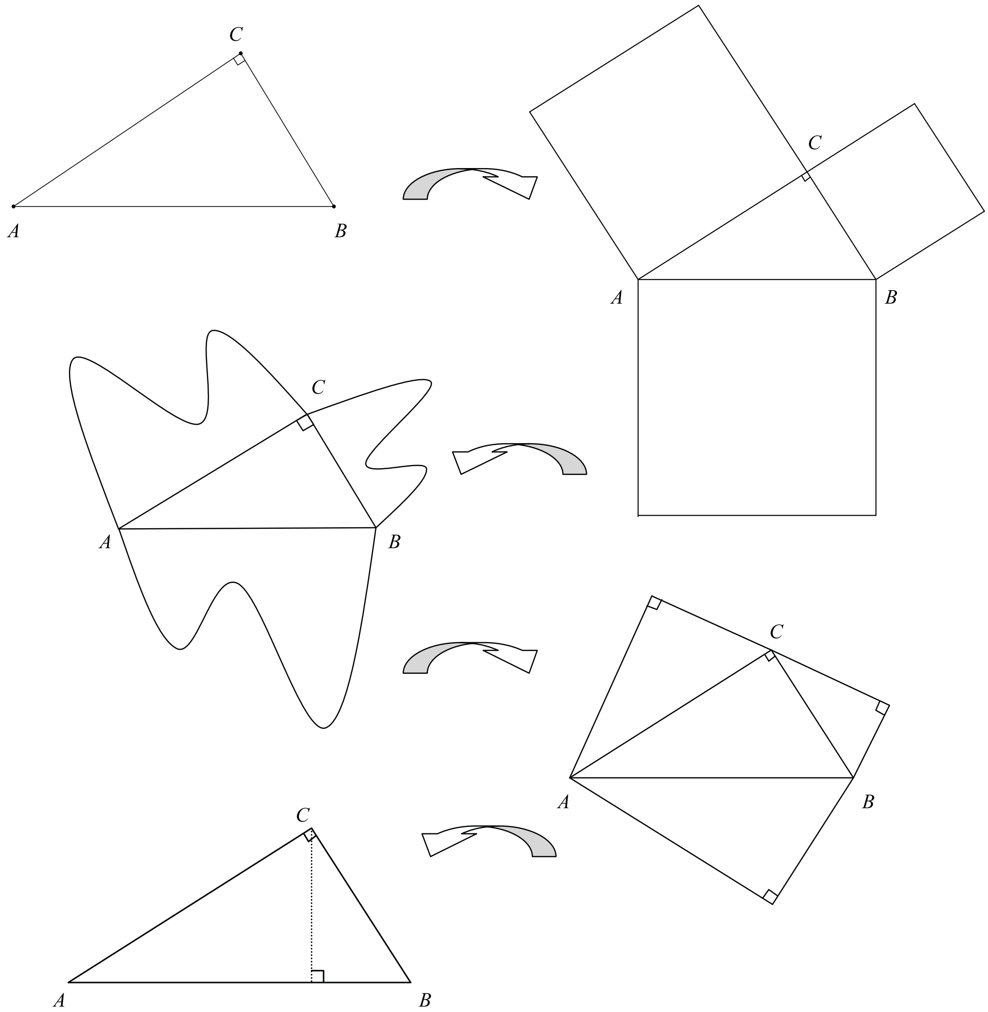

Figure 9

The solution to the famous and beautiful problem of finding the area of the lunes of Hippocrates also follows as an immediate corollary. That is, find the sum of the areas of the two shaded moon-shaped figures as shown in Figure 10, where the arcs on the sides of the triangle are semicircles.

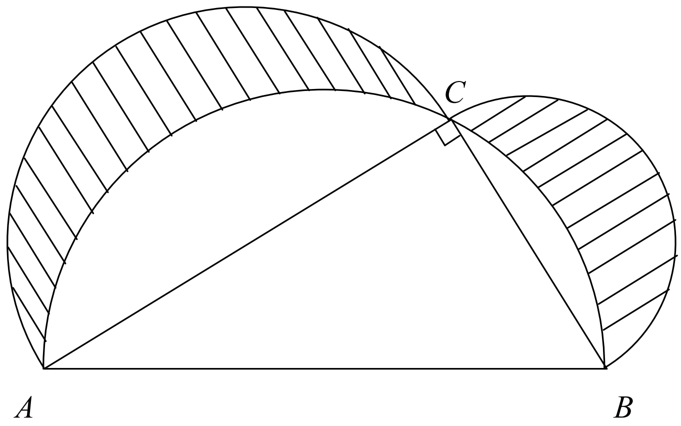

Figure 10

A concluding note, and a tetrahedron puzzle

The solutions to these eight problems present arguments using various aspects of the fundamental concept of symmetry: rotational symmetry, reflection (in several senses of the word), the symmetry of proportion, symmetry in an abstract sense (for example, in the conics problem), and asymmetry. The cognitive abilities called on in these arguments involve each of Krutetskii's "mathematical abilities."

The ability for students to exercise these abilities should be encouraged, fostered and developed by teachers of mathematics. Approaching solutions to problems in non-traditional ways, such as the ones in this article, should be modelled in learning activities. Equally, the teacher of mathematics is able to assess mathematical aptitude through problem solving activities that respond to these abilities. A facility to express Krutetskii's abilities by a student should be a sign of high cognitive potential, which should imply an appropriate educational response.

For fun, the reader is left with the following puzzle. There has been enough talk of symmetry to hint at approaches to a solution, but if difficulties do arise another hint might be to exercise your mathematical ability to reverse logical thinking.

Reproduce or photocopy the figure shown in Figure 11, and cut out the rectangle. It is important for your rectangle to have the same proportions as the one in the diagram. The idea is to fold the rectangle to make a tetrahedron, with no overlaps and with no gaps.

Figure 11

I wish to acknowledge the help of Mr John Smith, Mr Ray Smith and Mr Mark Ide in preparing the diagrams for this article.

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